# Minimal arithmetic thickness connecting discrete planes

Damien Jamet, Jean-Luc Toutant

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**Abstract**

While connected arithmetic discrete lines are entirely characterized, only partial results exist for the more general case of arithmetic discrete hyperplanes. In the present paper, we focus on the 3-dimensional case, that is on arithmetic discrete planes. Thanks to arithmetic reductions on a vector $\mathbf{n}$, we provide algorithms either to determine whether a given arithmetic discrete plane with $\mathbf{n}$ as normal vector is connected, or to compute the minimal thickness for which an arithmetic discrete plane with normal vector $\mathbf{n}$ is connected.

**Keywords :** Discrete geometry, arithmetic discrete planes, connectedness

## 1 Introduction

The discrete geometry attempts to provide an analogue of Euclidean geometry for the discrete space $\mathbb{Z}^n$. Such an investigation has not only theoretical motivations, but also practical applications since digital images can be seen as arrays of pixels.

In [1], J.-P. Reveillès initiated a new approach for linear discrete objects and defined arithmetic discrete lines as sets of pair of integers satisfying a double Diophantine inequality. The *arithmetic discrete line* with *normal vector* $\mathbf{n} \in \mathbb{R}^2$, *translation parameter* $\mu \in \mathbb{R}$ and *arithmetic thickness* $\omega \in \mathbb{R}$ is the set $\mathbb{L}(\mathbf{n}, \mu, \omega)$ defined by:

$$\mathbb{L}(\mathbf{n}, \mu, \omega) = \left\{ \mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2;\ 0 \leq l_{\mathbf{n},\mu}(\mathbf{v}) < \omega \right\},$$

where $l_{\mathbf{n},\mu}(\mathbf{v}) = n_1 v_1 + n_2 v_2 + \mu$.

Geometrically, it can be viewed as the set of integer points in a strip bounded by two parallel Euclidean lines. The width of this strip is ruled by the *arithmetic*

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*thickness*, which plays a key role in the definition. In particular, the connectedness of a given arithmetic discrete line is entirely characterized by its arithmetic thickness.

The definition of arithmetic discrete lines extends naturally to the definition of *arithmetic discrete planes* in the 3-dimensional discrete space $\mathbb{Z}^3$, and to the definition of *arithmetic discrete hyperplanes* in higher dimensions [2].

It is thus natural to expect a deep relation between the connectedness of an arithmetic discrete hyperplane and its arithmetic thickness. In fact, the 2-dimensional case is somewhat confusing since a connected arithmetic discrete line is also a separating set (see Section 2). Indeed, it is easy to characterized this last property for discrete planes with the arithmetic thickness, whereas it is yet unclear under which kind of conditions such a discrete object is connected.

Connectedness is a main topological property for the characterization and the understanding of discrete objects. Discrete planes are fundamental primitives in volume modelling. Improving the knowledge on connected discrete planes is thus of wide interest from theoretical perspective and may also lead to new powerful tools and applications. Besides, a section is devoted to this problem in [3]

In the present paper, we deal with the following questions:

1. Given $\mathbf{n} \in \mathbb{R}^3$, $\mu \in \mathbb{R}$ and $\omega \in \mathbb{R}$, is $\mathbb{P}(\mathbf{n}, \mu, \omega)$ connected?

2. Given $\mathbf{n} \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$, how much is the thickness of the thinnest connected arithmetic discrete plane with normal vector $\mathbf{n}$ and translation parameter $\mu$?

These questions have already been addressed. In [2], E. Andres, R. Acharya and C. Sibata characterized separating arithmetic discrete hyperplanes as connected set. This result gave a partial answer to the first question. In [4], Y. Gérard deeper investigated it. He provided an algorithm which determines whether a rational arithmetic discrete hyperplane, that is, with a normal vector $\mathbf{n} \in \mathbb{Z}^d$, is connected. He reduced the (possibly) infinite graph of connectedness of the considered arithmetic discrete hyperplane to a finite one by quotienting it by a subgroup of the lattice of periods of the arithmetic discrete hyperplane. In [5], V. Brimkov and R. Barneva focused on the second question. They introduced explicit formulas for some particular cases and provided an algorithm for the general case. Unfortunately, their algorithm appears to be incorrect [6].

In the present paper, we extend previous work [6] and give short and elementary algorithms, which take a vector $\mathbf{n} \in \mathbb{Z}^3$ as input and answer to both previously mentioned questions: to determine whether an arithmetic discrete plane is connected and to compute the minimal thickness making it connected. While Y. Gérard, V. Brimkov and R. Barneva approaches need to determine a connected component, our algorithms are *entirely* arithmetic and do not need to consider any graph of connectedness.

## 2 Basic Notions and First Properties

The aim of this section is to introduce the basic notions and definitions we use throughout the present paper.

Let $d$ be an integer equal or greater than 2 and let $\{\mathbf{e_1}, \dots, \mathbf{e_d}\}$ denote the canonical basis of the Euclidean vector space $\mathbb{R}^d$. Let us call *discrete set*, any subset of the *discrete space* $\mathbb{Z}^d$. In the following, for the sake of clarity, we denote by $(x_1, \dots, x_d)$ the point $\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e_i} \in \mathbb{R}^d$. An integer point $\mathbf{v} \in \mathbb{Z}^d$ is called a *voxel* (or a *pixel* if $d = 2$) and a subset of $\mathbb{Z}^d$,a *discrete set*.

**Definition 1** ($\kappa$-adjacency)**.** *Let* $\kappa \in \{0, \dots, d-1\}$*. Two voxels* $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ *are said to be* $\kappa$-adjacent *if:*

$$\|\mathbf{v} - \mathbf{w}\|_\infty = 1 \textit{ and } \|\mathbf{v} - \mathbf{w}\|_1 \leq d - \kappa.$$

**Remarque 1.** $\|\mathbf{v}\|_\infty = \max_{i \in \{1,\dots,d\}} \{|v_i|\}$ *and* $\|\mathbf{v}\|_1 = \sum_{i=1}^{d} |v_i|$.

In other words, the voxel $\mathbf{v}$ and the voxel $\mathbf{w}$ are $\kappa$*-adjacent* if they are distinct, the differences of their coordinates are at most 1 and $\mathbf{v}$ and $\mathbf{w}$ have at most $d - \kappa$ different coordinates. A $\kappa$*-path* is a (finite or infinite) sequence of consecutive $\kappa$-adjacent voxels. If $(\gamma_i)_{1 \leq i \leq n}$ is a finite $\kappa$-path, then we say that $\gamma$ *links* the voxel $\gamma_1$ to the voxel $\gamma_n$.

**Definition 2** ($\kappa$-connected sets)**.** *Let $E$ be a discrete set and let* $\kappa \in \{0, \dots, d-1\}$*. Then $E$ is* $\kappa$-connected *if, for each pair of voxels* $(\mathbf{v}, \mathbf{w}) \in E^2$*, there exists a $\kappa$-path in $E$ linking* $\mathbf{v}$ *to* $\mathbf{w}$*.*

In [1], J.-P. Reveillès introduced the arithmetic discrete line as a set of integer points satisfying a double Diophantine inequality.

**Definition 3** (Arithmetic discrete lines [1])**.** *Let* $\mathbf{n} \in \mathbb{R}^2$*,* $\mu \in \mathbb{R}$ *and* $\omega \in \mathbb{R}$*. The* arithmetic discrete line $\mathbb{L}(\mathbf{n}, \mu, \omega)$ *with* normal vector $\mathbf{n}$*,* translation parameter $\mu$ *and* arithmetic thickness $\omega$ *is the discrete set defined by:*

$$\mathbb{L}(\mathbf{n}, \mu, \omega) = \left\{\mathbf{v} \in \mathbb{Z}^2; 0 \leq l_{\mathbf{n},\mu}(\mathbf{v}) < \omega\right\}, \tag{1}$$

*where* $l_{\mathbf{n},\mu}(\mathbf{v}) = n_1 v_1 + n_2 v_2 + \mu$*.*

This definition extends in a natural way to higher dimensions:

**Definition 4** (Arithmetic discrete hyperplanes [1, 2])**.** *Let* $\mathbf{n} \in \mathbb{R}^d$*,* $\mu \in \mathbb{R}$ *and* $\omega \in \mathbb{R}$*. The* arithmetic discrete hyperplane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ *with* normal vector $\mathbf{n}$*,* translation parameter $\mu$ *and* arithmetic thickness $\omega$ *is the discrete set defined by:*

$$\mathbb{P}(\mathbf{n}, \mu, \omega) = \left\{\mathbf{v} \in \mathbb{Z}^d; 0 \leq p_{\mathbf{n},\mu}(\mathbf{v}) < \omega\right\}, \tag{2}$$

*where* $p_{\mathbf{n},\mu}(\mathbf{v}) = \mu + \sum_{i=1}^{d} n_i v_i$*.*

**Definition 5** (Rational arithmetic discrete hyperplanes)**.** *If there exists $\alpha \in \mathbb{R}^*$ such that $\alpha\mathbf{n} \in \mathbb{Z}^d$, then the arithmetic discrete hyperplane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ and its normal vector $\mathbf{n}$ are said to be* rational.

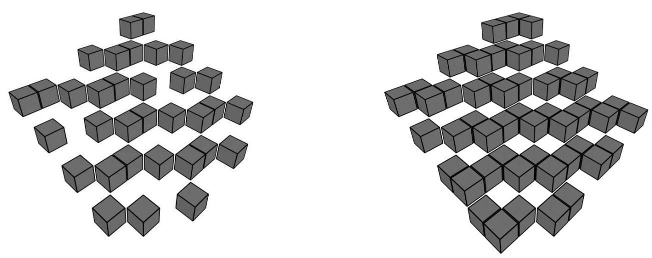
Throughout the present paper, if $\mathbb{P}(\mathbf{n}, \mu, \omega)$ is a rational arithmetic hyperplane, then we assume, without loss of generality, $\mathbf{n} \in \mathbb{Z}^d$, $\mu \in \mathbb{Z}$, $\omega \in \mathbb{Z}$ and $\gcd\{n_1, \ldots, n_d\} = 1$ [2].

In [1], J.-P. Reveillès showed how the $\kappa$-connectedness of an arithmetic discrete line only depends on its normal vector and its thickness:

**Theorem 1** ([1])**.** *Let $\mathbf{n} \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ and $\omega \in \mathbb{R}$. The arithmetic discrete line $\mathbb{L}(\mathbf{n}, \mu, \omega)$ is 0-connected (resp. 1-connected) if and only if $\omega \geq \|\mathbf{n}\|_\infty$ (resp. $\omega \geq \|\mathbf{n}\|_1$).*

It becomes natural to try to extend Theorem 1 to higher dimensions, that is, given $\mathbf{n} \in \mathbb{R}^d$, $\mu \in \mathbb{R}$ and $\kappa \in \{0, \ldots, d-1\}$, to try to characterize the thickness of the thinnest $\kappa$-connected arithmetic discrete hyperplane with normal vector $\mathbf{n}$ and translation parameter $\mu$.

Nevertheless, it is not difficult to exhibit a 0-connected arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ with $\omega < \|\mathbf{n}\|_\infty$ (see Fig. 1). Similarly, one easily finds a 2-connected arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ with $\omega < \|\mathbf{n}\|_1$.



(a) A 0-connected arithmetic discrete plane ($\omega = 11$) (b) A 1-connected arithmetic discrete plane ($\omega = 21$)

Figure 1: Connected arithmetic discrete planes (with normal vector $\mathbf{n} = (9, 14, 31)$) thinner than the naive one.

In fact, the case of arithmetic discrete lines is somewhat confusing. Indeed, in $\mathbb{Z}^2$, an arithmetic discrete line is $\kappa$-connected if, and only if, it separates (in some sense to define) the discrete space $\mathbb{Z}^2$. Let us introduce the notion of $\kappa$-separating sets.

**Definition 6** ($\kappa$-separating sets)**.** *Let $E$ be a discrete set and $\kappa \in \{0, \ldots, d-1\}$. Then $E$ is said to be $\kappa$*-separating *in $\mathbb{Z}^d$ if its complement in $\mathbb{Z}^d$ has exactly two $\kappa$-connected components.*

It directly follows from Theorem 1:

**Corollary 1.** *Let $\mathbf{n} \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ and $\omega \in \mathbb{R}$. The arithmetic discrete line $\mathbb{L}((\mathbf{n}, \mu, \omega)$ is 0-separating (resp. 1-separating) if and only if $\omega \geq \|\mathbf{n}\|_1$ (resp. $\omega \geq \|\mathbf{n}\|_\infty$).*

Although Theorem 1 does not seem to extend naturally to higher dimensions, there exists a quite nice extension of Corollary 1 relating to the $\kappa$-separating arithmetic discrete hyperplanes.

For the sake of clarity, let us introduce a notation, providing a norm on $\mathbb{R}^d$.

**Definition 7** ($\kappa$-minimality norms). *Let $\mathbf{x} \in \mathbb{R}^d$, $\kappa \in \{0, \ldots, d-1\}$ and let $\sigma$ be a permutation over the set $\{1, \ldots, d\}$ satisfying: $\forall i \in \{1, \ldots, d-1\}$, $|x_{\sigma(i)}| \leq |x_{\sigma(i+1)}|$. The* $\kappa$-minimality norm $]\mathbf{x}[_\kappa$ *of $\mathbf{x}$ is:*

$$]\mathbf{x}[_\kappa = \sum_{i=d-\kappa}^{d} |x_{\sigma(i)}|.$$

In other words, $]\mathbf{x}[_\kappa$ is equal to the sum of the $(\kappa+1)$ greatest absolute values of the coordinates of $\mathbf{x}$. One easily checks that, for each $\kappa \in \{0, \ldots, d-1\}$, the map $] \cdot [_\kappa : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a norm on $\mathbb{R}^d$. Moreover, one has $]\mathbf{x}[_0 = \|\mathbf{x}\|_\infty$ and $]\mathbf{x}[_{d-1} = \|\mathbf{x}\|_1$.

In $\mathbb{Z}^2$, the $\kappa$-connected arithmetic discrete lines are exactly the $(2-(\kappa+1))$-separating ones and Corollary 1 extends in any dimension $d \geq 2$ as follows:

**Theorem 2** ($\kappa$-separating hyperplanes [2]). *Let $\mathbf{n} \in \mathbb{R}^d$, let $\mu \in \mathbb{R}$ and let $\omega \in \mathbb{R}$. Let $\kappa \in \{0, \ldots, d-1\}$. The arithmetic discrete hyperplane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ is $\kappa$-separating in $\mathbb{Z}^d$ if and only if $\omega \geq ]\mathbf{n}[_\kappa$.*

Let us now notice that the fact that an arithmetic discrete hyperplane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ $\kappa$-separates $\mathbb{Z}^d$ does not depend on the translation parameter $\mu$.

Moreover, one shows that, given two rational arithmetic discrete hyperplanes $\mathbb{P}$ and $\mathbb{P}'$ with the same normal vector $\mathbf{n} \in \mathbb{Z}^d$, then $\mathbb{P}$ is the image of $\mathbb{P}'$ by a translation: $\mu$ can be expressed as an integer linear combination of the coordinates of $\mathbf{n}$, thanks to Bezout's Lemma. Hence, for all $\kappa \in \{0, \ldots, d-1\}$, the $\kappa$-connectedness of a rational arithmetic discrete hyperplane $\mathbb{P}(\mathbf{n}, \mu, \omega)$ does not depend on $\mu$. In other words, $\mathbb{P}(\mathbf{n}, \mu, \omega)$ is $\kappa$-connected if and only if so is $\mathbb{P}(\mathbf{n}, 0, \omega)$. In the general case, one does not know whether the $\kappa$-connectedness of an arithmetic discrete hyperplane depends on its translation parameter. [7] gives an idea of current knowledge on planes with arbitrary real coefficients.

Consequently, although it only provides a partial answer of our problem, we focus on the present paper on the arithmetic discrete planes with null translation parameter. For the sake of clarity, we refer to them by $\mathbb{P}(\mathbf{n}, \omega)$ and to the map $p_{\mathbf{n},0}$ by $p_{\mathbf{n}}$.

In the following sections, we investigate the three classes of connected arithmetic discrete planes. Let us start with the easiest case, that is, the 2-connected arithmetic discrete planes. The other cases can be deduced from that one.

# 3 $2$-connected arithmetic discrete planes

Let us first introduce some technical properties.

## 3.1 Technical properties

**Lemma 1** (A lower bound)**.** *Let* $\mathbf{n} \in \mathbb{R}^3$ *and* $\omega \in \mathbb{R}$*. If the arithmetic discrete plane* $\mathbb{P}(\mathbf{n}, \omega)$ *is* 2*-connected, then* $\omega \geq \|\mathbf{n}\|_\infty$*.*

**Proof** Without loss of generality, let us suppose that $0 \leq n_1 \leq n_2 \leq n_3$. If $n_1 = n_2 = 0$, then $n_3 = 1$ and $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected if and only if $\omega \geq 1$.
Assume now $n_2 \neq 0$ and let $0 < \omega < n_3$. Then $\mathbb{P}(\mathbf{n}, \omega)$ has voxels belonging to different level lines (i.e. voxels $\mathbf{v}$ and $\mathbf{w}$ such that $v_3 \neq w_3$). Let $\mathbf{v} \in \mathbb{P}(\mathbf{n}, \omega)$. Then, $0 \leq p_{\mathbf{n}}(\mathbf{v})$ and $w < n_3 \leq p_{\mathbf{n}}(\mathbf{v} + \mathbf{e_3}) = p_{\mathbf{n}}(\mathbf{v}) + n_3$. Hence, $\mathbf{v} + \mathbf{e_3} \notin \mathbb{P}(\mathbf{n}, \omega)$. In other words, two voxels of $\mathbb{P}(\mathbf{n}, \omega)$ belonging to different level line are not linked by a 2-path included in $\mathbb{P}(\mathbf{n}, \omega)$. Hence $\mathbb{P}(\mathbf{n}, \omega)$ is not 2-connected.

This bound is not very accurate but useful to prove that the arithmetic thicknesses of the arithmetic discrete planes, with a given normal vector, form an interval.

**Lemma 2.** *Let* $\mathbf{n} \in \mathbb{R}^3$*. The set* $\{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega)$ *is* 2*-connected*$\}$ *is an interval.*

**Proof** Without loss of generality, let us suppose that $0 \leq n_1 \leq n_2 \leq n_3$. Let $\omega \in \mathbb{R}$ such that $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected. According to Lemma 1, $\omega \geq n_3$. Let $\alpha \in \mathbb{R}_+$ and let $\mathbf{v} \in \mathbb{Z}^3$ such that $\beta = p_{\mathbf{n}}(\mathbf{v}) \in [\omega, \omega + \alpha[$. Let $(q, r) \in \mathbb{N} \times \mathbb{R}_+$ satisfying $\beta - \omega = qn_3 + r$ and $0 \leq r < n_3$. For all $k \in \{0, \ldots, q\}$, $p_{\mathbf{n}}(\mathbf{v} - k\mathbf{e_3}) = \beta - kn_3 \in [\omega, \omega + \alpha[$ and $p_{\mathbf{n}}(\mathbf{v} - (q+1)\mathbf{e_3}) \in [0, \omega[$. Hence, we have built a 2-path $(\mathbf{v} - k\mathbf{e_3})_{k \in \{0,\ldots,q+1\}}$ in $\mathbb{P}(\mathbf{n}, \omega + \alpha)$ linking $\mathbf{v}$ to the voxel $\mathbf{v} - (q+1)\mathbf{e_3} \in \mathbb{P}(\mathbf{n}, \omega)$ which is 2-connected by assumption.

Lemma 2 reduces the determination of $\{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega)$ *is* 2*-connected*$\}$ to the determination of its lower bound. Let us now define the minimal 2-connecting thickness of a vector.

**Definition 8** (Minimal 2-connecting thickness)**.** *Let* $\mathbf{n} \in \mathbb{R}^3$*. The* minimal 2-connecting thickness *of* $\mathbf{n}$ *is the number* $\Omega_2(\mathbf{n})$ *defined by:*

$$\Omega_2(\mathbf{n}) = \inf \{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega) \textit{ is } 2\textit{-connected}\}.$$

Let us remind that, if $\mathbf{n}$ is a rational vector then the thickness $\omega$ is considered to be an integer. In that case, $\Omega_2(\mathbf{n})$ becomes:

$$\Omega_2(\mathbf{n}) = \min \{\omega \in \mathbb{Z};\, \mathbb{P}(\mathbf{n}, \omega) \text{ is 2-connected}\}.$$

In this particular case, one remarks that $\mathbb{P}(\mathbf{n}, \Omega_2(\mathbf{n}))$ is 2-connected. As far as we know, it has not been proved it holds in the general case.

### 3.2 Arithmetic reductions preserving 2-connected components

In the present section, we show that the determination of $\Omega_2(\mathbf{n})$ reduces to the one of $\Omega_2(\mathbf{m})$, with $\|\mathbf{m}\|_\infty < \|\mathbf{n}\|_\infty$, by an elementary reduction on the components of $\mathbf{n}$.

**Theorem 3.** *Let $\mathbf{n} \in \mathbb{R}^3_+$ such that $0 \leq n_1 \leq n_2 \leq n_3$ and let $\mathbf{m} \in \mathbb{R}^3$ such that $\mathbf{m} = (n_1, n_2 + n_1, n_3 + n_1)$. For all $\omega \in \mathbb{R}$, the arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected if and only if so is $\mathbb{P}(\mathbf{m}, \omega + n_1)$.*

**Proof** Let us consider the map $\Psi_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by:

$$\begin{array}{cccccc} \Psi_2 & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} x_1 - x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \end{array}$$

One checks that $\Psi_2$ provides a bijection from $\mathbb{P}(\mathbf{n}, \omega)$ to $\mathbb{P}(\mathbf{m}, \omega)$.

1. Let us assume $\mathbb{P}(\mathbf{n}, \omega)$ to be 2-connected and let us show that $\mathbb{P}(\mathbf{m}, \omega + n_1)$ is 2-connected too. One first notes that, given an element $\mathbf{v} \in \mathbb{P}(\mathbf{m}, \omega + n_1)$, if $p_{\mathbf{m}}(\mathbf{v}) \in [\omega, \omega + a[$, then $\mathbf{v} - \mathbf{e_1} \in \mathbb{P}(\mathbf{m}, \omega + a)$ and $p_{\mathbf{m}}(\mathbf{v} - \mathbf{e_1}) \in [0, \omega[$. In other words, an element of $\mathbb{P}(\mathbf{m}, \omega + n_1)$ is either an element of $\mathbb{P}(\mathbf{m}, \omega)$ or 2-adjacent to an element of $\mathbb{P}(\mathbf{m}, \omega)$. Thanks to this remark, it remains to show that each pair of points of $\mathbb{P}(\mathbf{m}, \omega)$ is 2-linked in $\mathbb{P}(\mathbf{m}, \omega + n_1)$.
   Since $\Psi_2 : \mathbb{P}(\mathbf{n}, \omega) \longrightarrow \mathbb{P}(\mathbf{m}, \omega)$ is a bijection, it remains to show that the images of two 2-adjacent elements of $\mathbb{P}(\mathbf{n}, \omega)$ are 2-linked in $\mathbb{P}(\mathbf{m}, \omega + n_1)$. For short, we give a geometric interpretation of the action of the map $\Psi_2$ on two 2-adjacent voxels of $\mathbb{P}(\mathbf{n}, \omega)$ (see Figure 2) : the right grey voxels are the images of the left grey ones, while the right white voxels are elements of $\mathbb{P}(\mathbf{m}, \omega + n_1)$ which allow us to 2-link the grey voxels. For ensuring the existence of such white voxels, we remind that the right grey ones belongs to $\mathbb{P}(\mathbf{m}, \omega)$ and, for any such voxels $\mathbf{v}$, the (white) voxel $\mathbf{v} + \mathbf{e_1}$ belongs to $\mathbb{P}(\mathbf{m}, \omega + n_1)$.

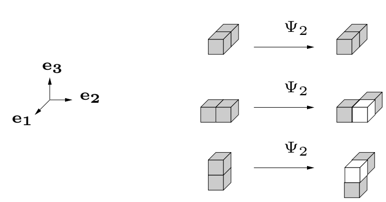



Figure 2: The action of $\Psi_2$ on two 2-adjacent voxels of $\mathbb{P}(\mathbf{n}, \omega)$.

2. Conversely, let us suppose $\mathbb{P}(\mathbf{m}, \omega + n_1)$ to be 2-connected. Then, $\omega + n_1 \geq \|\mathbf{n}\|_\infty$ (see Lemma 1). Hence, for all $\mathbf{v} \in \mathbb{P}(\mathbf{m}, \omega + n_1)$ such that $p_{\mathbf{m}}(\mathbf{v}) \in$

$[\omega, \omega + n_1[$, $p_{\mathbf{m}}(\mathbf{v} - \mathbf{e_1}) = p_{\mathbf{m}}(\mathbf{v}) - n_1 \in [0, \omega[$ and $\mathbf{v} - \mathbf{e_1} \in \mathbb{P}(\mathbf{m}, \omega)$. Let $\widetilde{\Psi}_2 : \mathbb{P}(\mathbf{m}, \omega + n_1) \longrightarrow \mathbb{P}(\mathbf{n}, \omega)$ be the surjective map defined by:

$$\begin{array}{rccc} \widetilde{\Psi}_2 & : & \mathbb{P}(\mathbf{m}, \omega + n_1) & \longrightarrow & \mathbb{P}(\mathbf{n}, \omega) \\ & & \mathbf{v} & \longmapsto & \left\{ \begin{array}{ll} \Psi_2^{-1}(\mathbf{v}), & \text{if } p_{\mathbf{m}}(\mathbf{v}) \in [0, \omega[, \\ \Psi_2^{-1}(\mathbf{v} - \mathbf{e_1}), & \text{otherwise.} \end{array} \right. \end{array}$$

Since $\widetilde{\Psi}_2$ is surjective, then it remains to show that the image of two 2-adjacent voxels in $\mathbb{P}(\mathbf{m}, \omega + n_1)$ are either equal or 2-linked in $\mathbb{P}(\mathbf{n}, \omega)$.

i. Let $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{P}(\mathbf{m}, \omega + n_1)$ such that $\mathbf{v} - \mathbf{w} = \mathbf{e_1}$. Two cases occur:
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [0, \omega[$, *i.e.* $\mathbf{v} \in \mathbb{P}(\mathbf{m}, \omega)$, then $\widetilde{\Psi}_2(\mathbf{v}) - \widetilde{\Psi}_2(\mathbf{w}) = \mathbf{e_1}$. Hence, $\widetilde{\Psi}_2(\mathbf{v})$ and $\widetilde{\Psi}_2(\mathbf{w})$ are 2-adjacent.
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [\omega, \omega + n_1[$, then $\widetilde{\Psi}_2(\mathbf{v}) = \widetilde{\Psi}_2(\mathbf{v} - \mathbf{e_1}) = \widetilde{\Psi}_2(\mathbf{w}) = \mathbf{e_1}$.

ii. Let $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{P}(\mathbf{m}, \omega + n_1)$ such that $\mathbf{v} - \mathbf{w} = \mathbf{e_2}$.
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [0, \omega[$, then $p_{\mathbf{m}}(\mathbf{v}) = p_{\mathbf{m}}(\mathbf{w}) + n_1 + n_2$ and, $\mathbf{v} + \mathbf{e_1} \in \mathbb{P}(\mathbf{m}, \omega + n_1)$ and $p_{\mathbf{m}}(\mathbf{w} + \mathbf{e_1}) \in [0, \omega[$ (since $n_1 \leq n_2$). One finally checks that $(\widetilde{\Psi}_2(\mathbf{w}), \widetilde{\Psi}_2(\mathbf{w} + \mathbf{e_1}), \widetilde{\Psi}_2(\mathbf{v}))$ forms a 2-path in $\mathbb{P}(\mathbf{n}, \omega)$.
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [\omega, \omega + n_1[$, then $\widetilde{\Psi}_2(\mathbf{v}) - \widetilde{\Psi}_2(\mathbf{w}) = \Psi_2^{-1}(\mathbf{e_2} - \mathbf{e_1}) = \mathbf{e_2}$.

iii. Let $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{P}(\mathbf{m}, \omega + n_1)$ such that $\mathbf{v} - \mathbf{w} = \mathbf{e_3}$.
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [0, \omega[$, then $p_{\mathbf{m}}(\mathbf{v}) = p_{\mathbf{m}}(\mathbf{w}) + n_1 + n_3$ and, $\mathbf{v} + \mathbf{e_1} \in \mathbb{P}(\mathbf{m}, \omega + n_1)$ and $p_{\mathbf{m}}(\mathbf{w} + \mathbf{e_1}) \in [0, \omega[$ (since $n_1 \leq n_3$). One then checks that $(\widetilde{\Psi}_2(\mathbf{w}), \widetilde{\Psi}_2(\mathbf{w} + \mathbf{e_1}), \widetilde{\Psi}_2(\mathbf{v}))$ form a 2-path in $\mathbb{P}(\mathbf{n}, \omega)$.
   - If $p_{\mathbf{m}}(\mathbf{v}) \in [\omega, \omega + n_1[$, then $\widetilde{\Psi}_2(\mathbf{v}) - \widetilde{\Psi}_2(\mathbf{w}) = \Psi_2^{-1}(\mathbf{e_3} - \mathbf{e_1}) = \mathbf{e_3}$.

$\square$

To sum up, let us give a geometrical interpretation of the last three cases (see Figure 3).

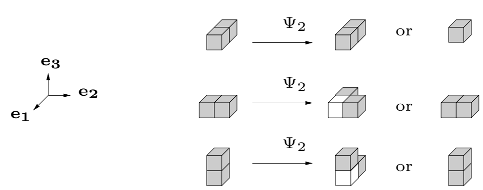



Figure 3: The action of $\widetilde{\Psi}_2$ on two 2-adjacent voxels of $\mathbb{P}(\mathbf{m}, \omega + n_1)$.

By induction, a direct consequence of Theorem 3 is:

**Corollary 2.** *Let $\mathbf{n} \in \mathbb{R}^3_+$ such that $0 \leq n_1 \leq n_2 \leq n_3$, let $q = \lfloor n_2/n_1 \rfloor$ and let $\mathbf{m} \in \mathbb{R}^3$ such that $\mathbf{m} = (n_1, n_2 - qn_1, n_3 - qn_1)$. For all $\omega \in \mathbb{R}$, the arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected if and only if so is $\mathbb{P}(\mathbf{m}, \omega + qn_1)$.*

The second important consequence of Theorem 3 links the minimal 2-connected thickness of $\mathbf{n}$ with the one of $\mathbf{m}$ (as defined in Corollary 2).

**Corollary 3.** *Let $\mathbf{n} \in \mathbb{R}^3_+$ such that $0 < n_1 \leq n_2 \leq n_3$ and $q = \lfloor n_2/n_1 \rfloor$. Then, we have:*

$$\Omega_2(n_1, n_2, n_3) = \Omega_2(n_1, n_2 - qn_1, n_3 - qn_1) + qn_1.$$

In this reduction, $n_1$, $n_2$ and $n_3$ are assumed to be all non-zero. If $n_1 = 0$, then

$$\mathbb{P}(\mathbf{n}, \omega) = \bigcup_{k \in \mathbb{Z}} \left( \left\{ \mathbf{v} \in \mathbb{Z}^3;\, x_1 = 0 \text{ and } 0 \leq p_{\mathbf{n}}(\mathbf{v}) < \omega \right\} + k\mathbf{e_1} \right)$$

and one checks that $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected if and only if so is the set $\left\{ \mathbf{v} \in \mathbb{Z}^3;\, x_1 = 0 \text{ and } 0 \leq p_{\mathbf{n}}(\mathbf{v}) < \omega \right\}$. From Theorem 1, it follows:

**Theorem 4.** *Let $\mathbf{n} \in \mathbb{R}^3_+$ such that $n_1 = 0$ and $\gcd\{n_2, n_3\} = 1$. Then, $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected if and only if $\omega \geq n_2 + n_3$.*

Composing both reductions (see Theorem 3 and Theorem 4) directly provides two algorithms: one that determines the minimal 2-connected thickness $\Omega(\mathbf{n})$ of a vector $\mathbf{n} \in \mathbb{N}^3$, one that returns whether an arithmetic discrete plane is 2-connected.

## 3.3 Applications

In the present section we provide an algorithm computing the 2-minimal thickness of a given integer vector $\mathbf{n}$. A first naive approach consists in "translating" Corollary 3, such as it is, in terms of an algorithm, in order to reduce $\mathbf{n}$ and increase $\omega$ while $\mathbf{n}$ does not satisfy conditions of Theorem 4. In fact, thanks to the following technical lemma, we can obtain a shorter and nicer way to compute $\Omega_2(\mathbf{n})$.

Notation. — Let $\phi_2 : \mathbb{Z}^\star \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3$ be the map defined by:

$$\phi_2(\mathbf{n}) = \left( n_2 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1, \min\left\{ n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1 \right\}, \max\left\{ n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1 \right\} \right).$$

Let $\Delta = \left\{ \mathbf{n} \in \mathbb{N}^3,\, 0 < n_1 \leq n_2 \leq n_3 \text{ and } \gcd\{n_1, n_2, n_3\} = 1 \right\}$.

**Lemma 3.** *Let $\mathbf{n} \in \Delta$ and let $\mathbf{n'} = \phi_2(\mathbf{n})$. Then, $0 \leq n'_1 \leq n'_2 \leq n'_3$ and $\gcd\{n'_1, n'_2, n'_3\} = 1$ (in particular, if $n'_1 \neq 0$ then $\mathbf{n'} \in \Delta$). Moreover,*

$$\|\mathbf{n}\|_1 - 2\Omega_2(\mathbf{n}) = \|\mathbf{n'}\|_1 - 2\Omega_2(\mathbf{n'}).$$

**Proof** The first assertions, that is, $0 \leq n'_1 \leq n'_2 \leq n'_3$ and $\gcd\{n'_1, n'_2, n'_3\} = 1$ are clear. The last assertions follows from that $\{n'_1, n'_2, n'_3\} = \left\{ n_1, n_2 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1 \right\}$ (see Notation above) and from that $\Omega_2(\mathbf{n'}) = \Omega_2(\mathbf{n}) - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1$. □

Given $\mathbf{n} \in \Delta$ and $\mathbf{n'} = \phi_2(\mathbf{n})$, one has $n'_1 < n_1$. Hence, there exists $k \in \mathbb{N}$ such that $\mathbf{n}^{(k)} = \phi_2^k = \underbrace{\phi_2 \circ \cdots \circ \phi_2}_{k \text{ times}}(\mathbf{n})$ satisfies $n_1^{(k)} = 0$. In that case, it follows from Theorem 4 that $\Omega_2(\mathbf{n'}) = n'_2 + n'_3$ and:

**Lemma 4.** *Let $\mathbf{n} \in \Delta$ and let $k \in \mathbb{N}$ such that $\mathbf{n'} = \phi_2^k = \underbrace{\phi_2 \circ \cdots \circ \phi_2}_{k \text{ times}}(\mathbf{n})$ satisfies $n'_1 = 0$. Then, $\Omega_2(\mathbf{n}) = \dfrac{\|\mathbf{n}\|_1 + \|\mathbf{n'}\|_1}{2}$.*

**Proof** By Lemma 3, $\|\mathbf{n}\|_1 - 2\Omega_2(\mathbf{n}) = \|\mathbf{n'}\|_1 - 2\Omega_2(\mathbf{n'})$. By Theorem 4, $\Omega_2(\mathbf{n'}) = \|\mathbf{n'}\|_1$ and the result follows. □

In fact, Lemma 3 and Lemma 4 prove the correction of the following algorithm calculating $\Omega_2(\mathbf{n})$ for any $\mathbf{n} \in \Delta$.

---

**Algorithm 1** Compute the minimal 2-connecting thickness

---

**Input :**
  $\mathbf{n} \in \mathbb{N}^3$ such that $0 \le n_1 \le n_2 \le n_3$ and $\gcd\{n_1, n_2, n_3\} = 1$
**Output :**
  $\Omega_2(\mathbf{n})$

  $\omega \leftarrow n_1 + n_2 + n_3$ ;
  **while** $(n_1 \neq 0)$ **do**
    $\mathbf{n} = \left(n_2 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1, \min\left\{n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1\right\}, \max\left\{n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1\right\}\right)$
    ;
  **end while**
  return $(\omega + n_2 + n_3)/2$ ;

---

This algorithm has obviously a constant space complexity, but also a reasonable time complexity.

**Proposition 1.** *Algorithm 1 runs in $O(log_2(n_2))$ time.*

**Proof** We refer to the vector $\mathbf{n}$ at iteration $i$ by $\mathbf{n}^{(i)}$. By definition of $q$, for all $i$, $n_1^{(i+1)} < n_2^{(i)}/2$. Moreover, since $n_2^{(i+1)} = \min\left\{n_1^{(i)}, n_3^{(i)} - \left\lfloor n_2^{(i)}/n_1^{(i)} \right\rfloor n_1\right\}$, one have $n_2^{(i+1)} \ge n_1^{(i)}$. Thus, by induction, for all $i$, $n_1^{(2i+1)} < n_2^{(0)}/2^i$. The algorithm ends when the value of $n_1^{(i)}$ is equal to 0. $n_1^{(i)}$ being an integer, this stopping criteria can be rewritten as $n_1^{(i)} < 1$. With previous statements, It means that $n_2^{(0)}/2^{i/2} < 1$ and the algorithm ends always after less than $2log_2(n_2^{(0)})$ iterations. □

The global computation of the thickness $\omega$ used in Algorithm 1 is not appropriate for the decision on the 2-connectedness of a given arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \omega)$. It rather requires to update this value at each iteration. Nevertheless Algorithm 2 is very similar to Algorithm 1. Indeed, we just have to initialize the process with the value of $\omega$ and, at each step, to decrease it according to Corollary 2. Then, $\mathbb{P}(\mathbf{n}, \omega)$ is 2-connected only if, at the end of the process, the value of $\omega$ remains greater than or equal to 0.

---

**Algorithm 2** Is a given arithmetic discrete plane 2-connected ?

---

**Input :**

$\omega \in \mathbb{N}^*$

$\mathbf{n} \in \mathbb{N}^3$ such that $0 \leq n_1 \leq n_2 \leq n_3$ and $\gcd\{n_1, n_2, n_3\} = 1$

**Output :**

Decision on the 2-connectedness of $\mathbb{P}(\mathbf{n}, \omega)$

**while** $(n_1 \neq 0$ and $\omega > 0)$ **do**

$$\mathbf{n} = \left(n_2 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1, \min\left\{n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1\right\}, \max\left\{n_1, n_3 - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1\right\}\right)$$

;

$$\omega \leftarrow \omega - \left\lfloor \frac{n_2}{n_1} \right\rfloor n_1 \;;$$

**end while**

return $(\omega > 0)$;

---

Comparison of Algorithm 2 with Y. Gérard's algorithm [4] is difficult since we do not know neither time complexity nor space complexity for this last one. In the one hand, it obviously requires space for storing the adjaceny graph and it uses set operations which are generally more time-consuming than simple arithmetic operations. In the other hand, it can apply whatever the dimension or the connectedness. Our algorithm just solve the case of 2-connected arithmetic discrete planes. In the sequel of the present paper, we extend it to the other connectedness of the 3-dimensional discrete space, but higher dimensions stay yet out of our scope.

# 4 1-connected arithmetic discrete planes

In the present section, we show how to reduce the problem of deciding the 1-connectedness of an arithmetic discrete plane to the 2-connectedness of another one.

Notation. — From now on, we denote by $\Psi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the following linear bijection:

$$\begin{array}{cccc} \Psi_1 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix} \end{array}$$

The following theorem shows how the map $\Psi_1$ is useful for the determination of the 1-connectedness of a given arithmetic discrete plane.

**Theorem 5.** *Let $\mathbf{n} \in \mathbb{R}^3$ such that $0 \leq n_1 \leq n_2 \leq n_3$ and let $\omega \in \mathbb{R}$. The arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \omega)$ is 1-connected if and only if the arithmetic discrete plane $\mathbb{P}(\Psi_1(\mathbf{n}), \omega)$ is 2-connected.*

Before proving this result and for the sake of clarity, let us introduce a terminology:

TERMINOLOGY. — Let $E$ be a discrete set, let $\mathbf{v}$ and $\mathbf{v'}$ be two elements of $E$ and let $\kappa \in \{0, 1, 2\}$. We say that $\mathbf{v}$ and $\mathbf{v'}$ are *$\kappa$-linked in $E$* if they share the same $\kappa$-connected component in $E$.

**Proof** For short, let us state $\mathbb{P} = \mathbb{P}(\mathbf{n}, \omega)$ and $\mathbb{P}' = \mathbb{P}(\Psi_1(\mathbf{n}), \omega)$. Let $\widetilde{\Psi}_1 = {}^t\Psi_1{}^{-1}$:

$$\begin{array}{cccc} \widetilde{\Psi}_1 & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{pmatrix} \end{array}$$

One easily checks that $\widetilde{\Psi}_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ provides a bijection from $\mathbb{P}$ to $\mathbb{P}'$ and, for all $\mathbf{v} \in \mathbb{P}$, $p_{\mathbf{n}}(\mathbf{v}) = p_{\Psi_1(\mathbf{n})}(\widetilde{\Psi}_1(\mathbf{v}))$.

It remains to show that two voxels $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{P}$ are 1-linked in $\mathbb{P}$ if and only if $\widetilde{\Psi}_1(\mathbf{v})$ and $\widetilde{\Psi}_1(\mathbf{w})$ are 2-linked in $\mathbb{P}'$. Equivalently, it is sufficient to prove the following assertions:

1. Let $\mathbf{v} \in \mathbb{P}$ and $\mathbf{w} \in \mathbb{P}$. If $\mathbf{v}$ and $\mathbf{w}$ are 1-adjacent in $\mathbb{P}$, then $\widetilde{\Psi}_1(\mathbf{v})$ and $\widetilde{\Psi}_1(\mathbf{w})$ are 2-linked in $\mathbb{P}'$.

2. Let $\mathbf{v'} \in \mathbb{P}'$ and $\mathbf{w'} \in \mathbb{P}'$. If $\mathbf{v'}$ and $\mathbf{w'}$ are 2-adjacent in $\mathbb{P}'$, then $\widetilde{\Psi}_1^{-1}(\mathbf{v'})$ and $\widetilde{\Psi}_1^{-1}(\mathbf{w'})$ are 1-linked in $\mathbb{P}$.

1. Let $\mathbf{v}$ and $\mathbf{w}$ be two 1-adjacent voxels of $\mathbb{P}$.

   i If $\mathbf{v} - \mathbf{w} = \mathbf{e_1}$, then $\widetilde{\Psi}_1(\mathbf{v}) - \widetilde{\Psi}_1(\mathbf{w}) = \mathbf{e_1}$. Hence $\widetilde{\Psi}_1(\mathbf{v})$ and $\widetilde{\Psi}_1(\mathbf{w})$ are 2-linked in $\mathbb{P}'$.

   ii If $\mathbf{v} - \mathbf{w} = \mathbf{e_2}$, then $\widetilde{\Psi}_1(\mathbf{v}) - \widetilde{\Psi}_1(\mathbf{w}) = \mathbf{e_1} + \mathbf{e_2}$. Moreover, one checks that $\widetilde{\Psi}_1(\mathbf{v}) + \mathbf{e_1} \in \mathbb{P}'$ and we have shown that $\widetilde{\Psi}_1(\mathbf{v})$ and $\widetilde{\Psi}_1(\mathbf{w})$ are 2-linked in $\mathbb{P}'$. Geometrically, the action of $\widetilde{\Psi}_1$ can be represented as in Figure 4 : the grey voxels correspond to the images of the extremities of the original 1-path, the white ones belong to $\mathbb{P}'$ and allow us to construct a 2-path in $\mathbb{P}'$.

   The other cases, that is, $\mathbf{v} - \mathbf{w} \in \{\mathbf{e_3}, \mathbf{e_1} \pm \mathbf{e_2}, \mathbf{e_2} \pm \mathbf{e_3}, \mathbf{e_1} \pm \mathbf{e_3}\}$ are handled by the same method. Moreover, the cases $\mathbf{v} - \mathbf{w} \in \{\mathbf{e_1} + \mathbf{e_2}, \mathbf{e_2} + \mathbf{e_3}, \mathbf{e_1} + \mathbf{e_3}\}$ can be splited into the elementary cases $\mathbf{v} - \mathbf{w} \in \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$.

2. Conversely, one checks that for all $\mathbf{v} \in \mathbb{Z}^3$ satisfying $\|\mathbf{v}\|_1 \leq 1$, then $\|\widetilde{\Psi}_1^{-1}(\mathbf{v})\|_\infty \leq 1$ and $\|\widetilde{\Psi}_1^{-1}(\mathbf{v})\|_\infty \leq 2$. It follows that the image under $\widetilde{\Psi}_1^{-1}$ of two 2-adjacent voxels of $\mathbb{P}'$ are 1-adjacent. $\square$

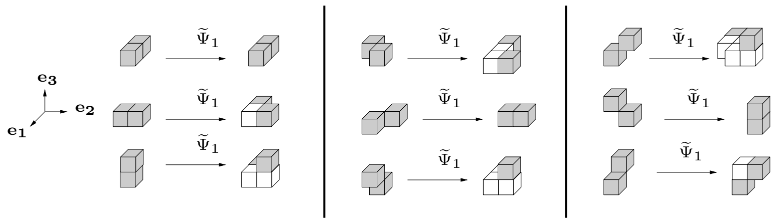



Figure 4: The action of $\widetilde{\Psi}_1$ on two 2-adjacent voxels of $\mathbb{P}$.

From Theorem 5 and Lemma 2, it directly follows that, given $\mathbf{n} \in \mathbb{R}^3$, the set $\{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega) \text{ is 1-connected}\}$ is an interval. Its determination is equivalent to the one of its lower bound, also called the minimal 1-connecting thickness of $\mathbf{n}$:

**Definition 9** (Minimal 1-connecting thickness)**.** *Let* $\mathbf{n} \in \mathbb{R}^3$*. The* minimal 1-connecting thickness *of* $\mathbf{n}$ *is the number* $\Omega_1(\mathbf{n})$ *defined by:*

$$\Omega_1(\mathbf{n}) = \inf \{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega) \textit{ is } 1\textit{-connected}\}.$$

A direct consequence of Theorem 5 is:

**Corollary 4.** *Let* $\mathbf{n} \in \mathbb{R}^3$ *such that* $0 \le n_1 \le n_2 \le n_3$ *and let* $\Psi_1$ *be as defined in* Notation *above. Then* $\Omega_1(\mathbf{n}) = \Omega_2(\Psi_1(\mathbf{n}))$*.*

Thanks to Algorithm 1 and Corollary 4, one easily determines the minimal 1-connecting thickness of a given rational vector. In the same way, one easily decides whether a given arithmetic discrete plane is 1-connected.

# 5 0-connected arithmetic discrete planes

The last case, which has been the most studied in the literature [2, 4, 5], concerns the 0-connectedness of the arithmetic discrete plane. This part is almost similar to the previous one. Let us first recall a technical lemma, called *Symmetry lemma* in [5]:

**Lemma 5** (Symmetry lemma [5])**.** *Let* $\mathbf{n} \in \mathbb{R}^3_+$ *such that* $0 \le n_1, n_2 \le n_3$ *and let* $\mathbf{n'} = (n_3 - n_1)\mathbf{e_1} + n_2\mathbf{e_2} + n_3\mathbf{e_3}$*. Let* $\omega \in \mathbb{R}$*. The arithmetic discrete plane* $\mathbb{P}(\mathbf{n}, \omega)$ *is* 0*-connected if and only so is* $\mathbb{P}(\mathbf{n'}, \omega)$*.*

The main interest of Lemma 5 is that it allows us to assume $0 \le 2n_1 \le n_3$. Moreover, since the role of $n_1$ and $n_2$ are symmetric in Lemma 5, one can also suppose $0 \le 2n_2 \le n_3$. In particular, one can suppose that $0 \le n_1 \le n_2 \le n_1 + n_2 \le n_3$.

Let us now introduce a notation before stating the main result of the present section.

Notation. — From now on, we denote by $\Psi_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the following linear bijection:

$$\begin{array}{cccc} \Psi_0 & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 - x_1 \end{pmatrix} \end{array}$$

The main result of this section is then:

**Theorem 6.** *Let $\mathbf{n} \in \mathbb{R}^3$ such that $0 \leq n_1 \leq n_2 \leq n_1 + n_2 \leq n_3$ and let $\omega \in \mathbb{R}$. The arithmetic discrete plane $\mathbb{P}(\mathbf{n}, \omega)$ is 0-connected if and only if the arithmetic discrete plane $\mathbb{P}(\Psi_0(\mathbf{n}), \omega)$ is 2-connected.*

**Proof** The proof is almost the same as the one of Theorem 5. Here we consider the action of the map $\widetilde{\Psi_0} = {}^t\Psi_0{}^{-1}$ from $\mathbb{P}(\mathbf{n}, \omega)$ to $\mathbb{P}(\Psi_0(\mathbf{n}), \omega)$. □

From Theorem 6 and Lemma 2, it directly follows that, given $\mathbf{n} \in \mathbb{R}^3$, the set $\{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, w) \text{ is 0-connected}\}$ is an interval. Its computation is equivalent to the one of its lower bound, also called the minimal 0-connecting thickness of $\mathbf{n}$:

**Definition 10** (minimal 0-connecting thickness)**.** *Let $\mathbf{n} \in \mathbb{R}^3$. The* minimal 0-connecting thickness *of $\mathbf{n}$ is the number $\Omega_0(\mathbf{n})$ defined by:*

$$\Omega_0(\mathbf{n}) = \inf \{\omega \in \mathbb{R};\, \mathbb{P}(\mathbf{n}, \omega) \textit{ is 0-connected}\}.$$

A direct consequence of Theorem 6 is:

**Corollary 5.** *Let $\mathbf{n} \in \mathbb{R}^3$ such that $0 \leq n_1 \leq n_2 \leq n_1 + n_2 \leq n_3$ and let $\Psi_0$ be as defined in* Notation *above. One has $\Omega_0(\mathbf{n}) = \Omega_2(\Psi_0(\mathbf{n}))$.*

Thanks to Algorithm 1, one easily determines the minimal 0-connecting thickness of a given rational vector. Just remind that, with no loss of generality, up to exchange $n_1$ and $n_3 - n_1$ (resp. $n_2$ and $n_3 - n_2$), one can suppose that $0 \leq n_1 \leq n_2 \leq n_1 + n_2 \leq n_3$ and apply Lemma 5 and Theorem 6 to determine $\Omega_0(\mathbf{n})$ in every case. In the same way, one can decide whether a given arithmetic discrete plane is 0-connected.

## 5.1 Additional remarks on the minimal 0-connected thickness

The determination of $\Omega_0(\mathbf{n})$ has already been deeply investigated in [5] as already mentioned. V. Brimkov and R. Barneva focused on rational arithmetic discrete planes and found explicit formulas in some particular cases:

**Theorem 7** (Explicit formulas [5])**.** *Let $\mathbf{n} \in \mathbb{Z}^3$ satisfying $0 \leq n_1 \leq n_2 \leq n_3$. One has:*

- *if* $n_3 < n_1 + n_2/2$, $\Omega_0(\mathbf{n}) = n_1 + n_2 - n_3 + \gcd(n_3 - n_2, n_3 - n_1) - 1$,
- *if* $n_1 + n_2 < n_3 < 2n_2 - n_1$, $\Omega_0(\mathbf{n})) = n_2 - n_1 + gcd(n_1, n_3 - n_2) - 1$,
- *if* $n_3 \geq 2n_2 + n_1$, $\Omega_O(\mathbf{n}) = n_3 - (n_1 + n_2) + \gcd(n_1, n_2) - 1$.

## Conclusion

In the present paper, we have shown how to compute the minimal 2-connecting thickness of a vector $\mathbf{n}$. The reduction exhibited in Theorem 3 works whatever the type (rational or irrational) of the considered vector. Nevertheless, we have restricted our investigation to rational vectors and provided an algorithm which computes their minimal 2-connecting thickness. This algorithm can be easily adapted to decide whether a given rational arithmetic discrete plane is 0-connected or not. Then, we have shown how to reduce the problem of determining the minimal 1-connecting and 0-connecting thicknesses of a vector $\mathbf{n}$ to the determination of the minimal 2-connecting thickness of an appropriate vector (see Theorem 5 and Theorem 6).

In forthcoming work, we plan to deeply investigate the case of non-rational arithmetic discrete planes. Since reductions of Theorem 3, Theorem 5 and Theorem 6 do not depend on the nature of the input vector (integer or not), we hope to extend our approach to any vector $\mathbf{n} \in \mathbb{R}^3$. Currently, some particular points still need to be investigated. In particular, given $\mathbf{n} \in \mathbb{R}^3$ and $\kappa \in \{0, 1, 2\}$, we do not know whether $\mathbb{P}(\mathbf{n}, \Omega_\kappa(\mathbf{n}))$ is $\kappa$-connected. In other words, is $\Omega_\kappa(\mathbf{n})$ the smallest element of the set $\{\omega \in \mathbb{R};\ \mathbb{P}(\mathbf{n}, \omega) \text{ is } \kappa\text{-connected}\}$ ? or just a lower bound? Besides, does the translation parameter change the $\kappa$-connectedness of an arithmetic discrete plane if its normal vector is not rational?

Another interesting investigation will be the extension of this work to arithmetic discrete hyperplane in any dimension.

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